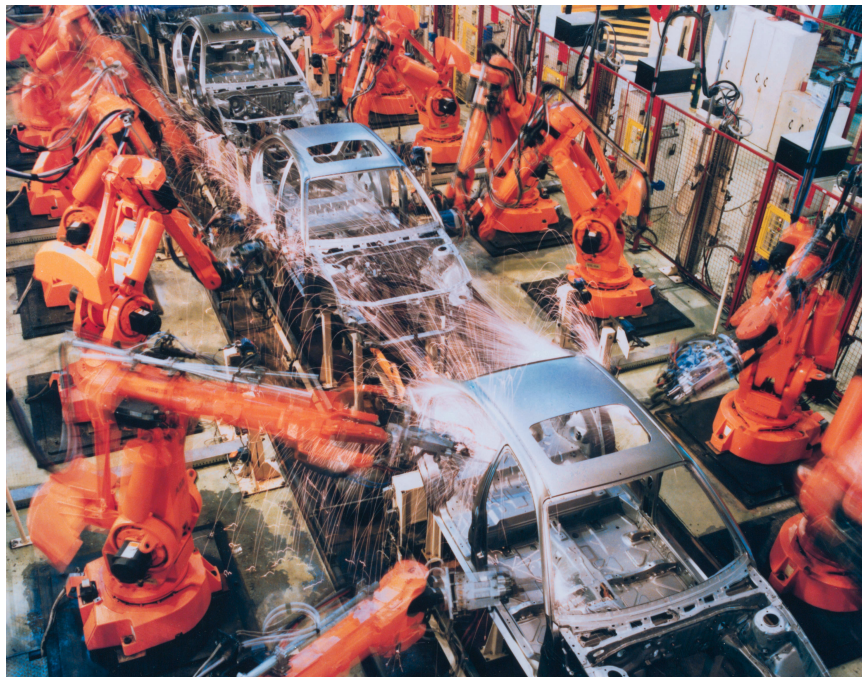


Robot Dynamics and Control

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Chapter 1

Rapid Review of Kinematics

1.1 Introduction

In this chapter we provide a brief review of basic kinematics that will be useful to derive the dynamics of robots in the next chapter and to understand the subsequent control chapters. We discuss rigid body motions, rotations and translations, described by so-called **Homogeneous Transformations**. These are used to describe the **Forward Kinematics**, both position and velocity kinematics. We also touch briefly on the problem of **Inverse Kinematics**. Finally, we will derive the manipulator **Jacobian** which gives the velocity kinematics.

We consider a robot manipulator with n —**links** interconnected by **joints** into a **kinematic chain**. Figure 1 shows a serial link (left) and a parallel link (right) manipulator. A parallel robot, by definition, contains two or more independent serial link chains. For simplicity we shall confine



Figure 1.1: A serial manipulator (left), the ABB IRB1400, and a parallel manipulator (right), the ABB IRB940Tricept. Photos courtesy of ABB Robotics.

our discussion to serial link manipulators with **revolute** or **prismatic** joints.

Figure 2 shows a six degree-of-freedom, revolute joint robot. We define the joint variables, q_1, \dots, q_n , as the relative angles between the links, for example, q_i is the angle between link i and link $i - 1$. A vector $q = (q_1, \dots, q_n)^T$ with each $q_i \in [0, 2\pi)$, is called a **configuration**. The set of all possible configurations is called **configuration space** or **joint space**, which we denote as \mathcal{C} . The configuration space for a revolute joint robot is an n -dimensional torus, $\mathcal{T}^n = S^1 \times \dots \times S^1$, where S^1 is the unit circle.

The **task space** is the space of all positions and orientations (called **poses**) of the end-effector. We attach a coordinate frame, called the **base frame**, or **world frame**, at the base of the robot and a second frame, called the **end-effector frame** or **task frame**, at the end-effector. The end-effector pose can then be described by specifying the amount of rotation and translation of the task frame relative to the base frame. We show how to do this in the next section.

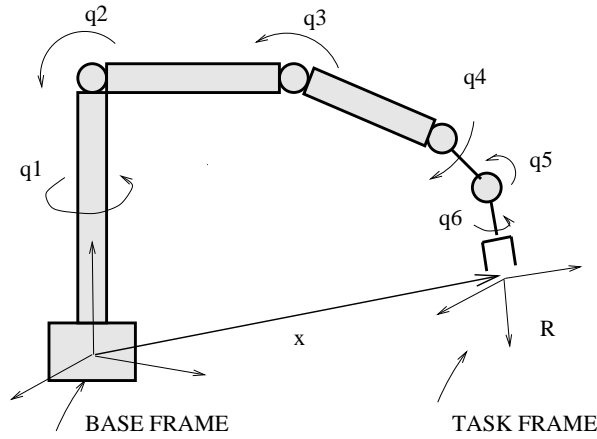


Figure 1.2: A Serial Link Manipulator showing the attached Base Frame, Task Frame, and configuration variables.

1.2 Rigid Motions and Homogeneous Transformations

In order to represent the relative position and orientation of one rigid body with respect to another, we can attach coordinate frames to each body, and then specify the geometric relationships between these coordinate frames.

Definition 1 *The Special Orthogonal Group of order 3, denoted $SO(3)$, is the set of 3×3 matrices, R satisfying*

$$R^T R = I \quad (1.1)$$

$$\det(R) = 1 \quad (1.2)$$

Elements of $SO(3)$ are called **Rotation Matrices**. It is easy to show that $SO(3)$ defines a **Non-Abelian Group**. It follows from equations (1) and (2) that the columns (respectively, the rows) of a rotation matrix are of unit length and mutually orthogonal. Referring to Figure 3, they are, in fact, the **direction cosines** of frame 1 relative to frame 0.

Example: 1.1 *Consider the frames shown in Figure 4. By inspection, we note that the x_1, y_1, z_1 axes are in the direction of the $-z_0, -x_0$, and y_0 axes, respectively. Therefore, the Rotation matrix,*

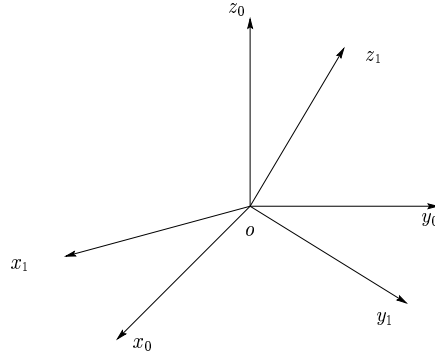


Figure 1.3: Two right hand coordinate frames with frame 1 rotated relative to frame 0

R_0^1 , relating these frames is given as

$$R_0^1 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \quad (1.3)$$

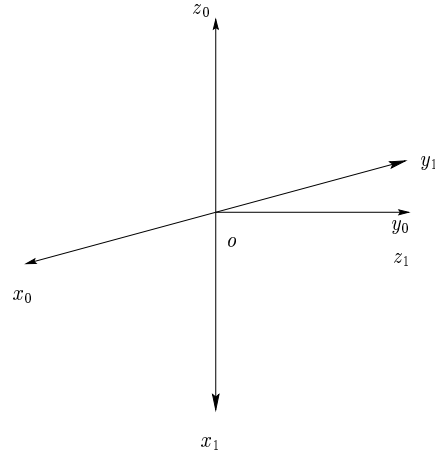


Figure 1.4: Example 0.1

1.2.1 Parameterizations of Rotations

The nine elements r_{ij} in a general rotational transformation R are not independent quantities. Indeed a rigid body possesses at most three rotational degrees-of-freedom and thus at most three quantities are required to specify its orientation. We define a set of basic rotation matrices, relative to the coordinate axes, as follows

$$R_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha \\ 0 & s_\alpha & c_\alpha \end{bmatrix}; \quad R_{y,\theta} = \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix}; \quad R_{z,\psi} = \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (1.4)$$

Any element of $SO(3)$ can then be represented as a product (non-uniquely) of basic rotation matrices. So-called **Euler Angles** and **Roll-Pitch-Yaw Angles** are common ways to specify a rotation matrix in terms of three independent quantities.

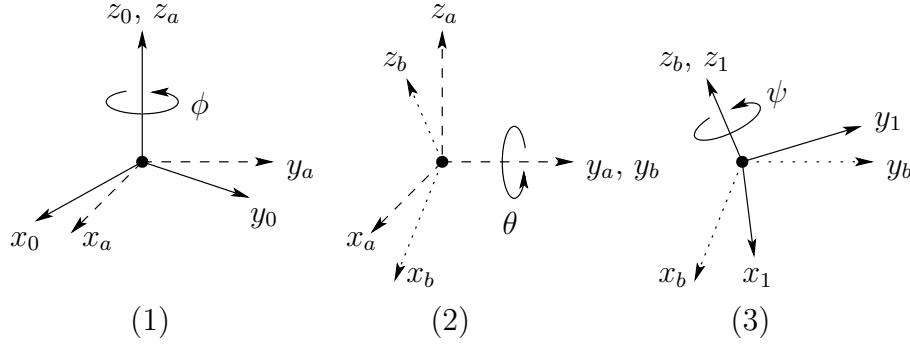


Figure 1.5: Euler angle representation.

Example: 1.2 (Euler Angles)

Consider the fixed coordinate frame $o_0x_0y_0z_0$ and the rotated frame $o_1x_1y_1z_1$ shown in Figure 5. We can specify the orientation of the frame $o_1x_1y_1z_1$ relative to the frame $o_0x_0y_0z_0$ by three angles (ϕ, θ, ψ) , known as Euler Angles, and obtained by three successive rotations as follows: First rotate about the z -axis by the angle ϕ . Next rotate about the current y -axis by the angle θ . Finally rotate about the current z -axis by the angle ψ . In Figure 5, frame $o_ax_ay_az_a$ represents the new coordinate frame after the rotation by ϕ , frame $o_bx_by_bz_b$ represents the new coordinate frame after the rotation by θ , and frame $o_1x_1y_1z_1$ represents the final frame, after the rotation by ψ . Frames $o_ax_ay_az_a$ and $o_bx_by_bz_b$ are shown in the figure only to help you visualize the rotations.

In terms of the basic rotation matrices the resulting rotational transformation R_1^0 can be generated as the product

$$R_1^0 = R_{z,\phi} R_{y,\theta} R_{z,\psi} \quad (1.5)$$

$$\begin{aligned}
 &= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}. \quad (1.6)
 \end{aligned}$$

1.2.2 Inverse Euler Angles

Suppose $U = (u_{ij}) \in SO(3)$ is given and R_1^0 is the Euler angle transformation (5). The problem then is to find the Euler angles ϕ, θ, ψ satisfying the matrix equation

$$\begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta & s_\theta s_\psi & c_\theta \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix}. \quad (1.7)$$

Suppose that not both of u_{13}, u_{23} are zero. Then the above equation shows that $s_\theta \neq 0$, and hence that not both of u_{31}, u_{32} are zero. If not both u_{13} and u_{23} are zero, then $u_{33} \neq \pm 1$, and we have $c_\theta = u_{33}$, $s_\theta = \pm \sqrt{1 - u_{33}^2}$ so

$$\theta = \text{Atan} \left(u_{33}, \sqrt{1 - u_{33}^2} \right) \quad (1.8)$$

or

$$\theta = A \tan \left(u_{33}, -\sqrt{1 - u_{33}^2} \right). \quad (1.9)$$

If we choose the first value for θ , then $s_\theta > 0$, and

$$\phi = A \tan(u_{13}, u_{23}) \quad (1.10)$$

$$\psi = A \tan(-u_{31}, u_{32}). \quad (1.11)$$

If we choose the second value for θ , then $s_\theta < 0$, and

$$\phi = A \tan(-u_{13}, -u_{23}) \quad (1.12)$$

$$\psi = A \tan(u_{31}, -u_{32}). \quad (1.13)$$

Thus there are two solutions depending on the sign chosen for θ .

If $u_{13} = u_{23} = 0$, then the fact that U is orthogonal implies that $u_{33} = \pm 1$, and that $u_{31} = u_{32} = 0$. Thus U has the form

$$U = \begin{bmatrix} u_{11} & u_{12} & 0 \\ u_{21} & u_{22} & 0 \\ 0 & 0 & \pm 1. \end{bmatrix} \quad (1.14)$$

If $u_{33} = 1$, then $c_\theta = 1$ and $s_\theta = 0$, so that $\theta = 0$. In this case (7) becomes

$$\begin{bmatrix} c_\phi c_\psi - s_\phi s_\psi & -c_\phi s_\psi - s_\phi c_\psi & 0 \\ s_\phi c_\psi + c_\phi s_\psi & -s_\phi s_\psi + c_\phi c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_{\phi+\psi} & -s_{\phi+\psi} & 0 \\ s_{\phi+\psi} & c_{\phi+\psi} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & 0 \\ u_{21} & u_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (1.15)$$

Thus the sum $\phi + \psi$ can be determined as

$$\begin{aligned} \phi + \psi &= A \tan(u_{11}, u_{21}) \\ &= A \tan(u_{11}, -u_{12}). \end{aligned} \quad (1.16)$$

Since only the sum $\phi + \psi$ can be determined in this case there are infinitely many solutions. We may take $\phi = 0$ by convention, and define ψ by (14). If $u_{33} = -1$, then $c_\theta = -1$ and $s_\theta = 0$, so that $\theta = \pi$. In this case (7) becomes

$$\begin{bmatrix} -c_{\phi-\psi} & -s_{\phi-\psi} & 0 \\ s_{\phi-\psi} & c_{\phi-\psi} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & 0 \\ u_{21} & u_{22} & 0 \\ 0 & 0 & -1 \end{bmatrix}. \quad (1.17)$$

The solution is thus

$$\phi - \psi = A \tan(-u_{11}, -u_{12}) = A \tan(-u_{21}, -u_{22}). \quad (1.18)$$

As before there are infinitely many solutions.

1.2.3 Homogeneous Transformations

Rigid motions includes translations in addition to rotation.

Definition 2 *The Special Euclidean Group of order 3, $SE(3)$, is then defined as*

$$SE(3) = SO(3) \times R^3 \quad (1.19)$$

$$= \{(R, r) \text{ where } R \in SO(3) \text{ and } r \in R^3\} \quad (1.20)$$

If we have the two rigid motions

$$p^0 = R_1^0 p^1 + d_1^0 \quad (1.21)$$

and

$$p^1 = R_2^1 p^2 + d_2^1 \quad (1.22)$$

then their composition defines a third rigid motion, which we can describe by substituting the expression for p^1 from (22) into (21)

$$p^0 = R_1^0 R_2^1 p^2 + R_1^0 d_2^1 + d_1^0. \quad (1.23)$$

Since the relationship between p^0 and p^2 is also a rigid motion, we can equally describe it as

$$p^0 = R_2^0 p^2 + d_2^0. \quad (1.24)$$

Comparing Equations (23) and (24) we have the relationships

$$R_2^0 = R_1^0 R_2^1 \quad (1.25)$$

$$d_2^0 = d_1^0 + R_1^0 d_2^1. \quad (1.26)$$

Equation (25) shows that the orientation transformations can simply be multiplied together and Equation (26) shows that the vector from the origin o_0 to the origin o_2 has coordinates given by the sum of d_1^0 (the vector from o_0 to o_1 expressed with respect to $o_0x_0y_0z_0$) and $R_1^0 d_2^1$ (the vector from o_1 to o_2 , expressed in the orientation of the coordinate system $o_0x_0y_0z_0$).

A comparison of this with the matrix identity

$$\begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2^1 & d_2^1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1^0 R_2^1 & R_1^0 d_2^1 + d_1^0 \\ 0 & 1 \end{bmatrix} \quad (1.27)$$

where 0 denotes the row vector $(0, 0, 0)$, shows that the rigid motions can be represented by the set of matrices of the form

$$H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}; R \in SO(3). \quad (1.28)$$

Using the fact that R is orthogonal it is an easy exercise to show that the inverse transformation H^{-1} is given by

$$H^{-1} = \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix}. \quad (1.29)$$

Transformation matrices of the form (28) are called **homogeneous transformations**. In order to represent a rigid motion by a matrix multiplication, one needs to augment the vectors p^0 and p^1 by the addition of a fourth component of 1 as follows. Set

$$P^0 = \begin{bmatrix} p^0 \\ 1 \end{bmatrix} \quad (1.30)$$

$$P^1 = \begin{bmatrix} p^1 \\ 1 \end{bmatrix}. \quad (1.31)$$

The vectors P^0 and P^1 are known as **homogeneous representations** of the vectors p^0 and p^1 , respectively. It can now be seen directly that the rigid motion transformation is equivalent to the (homogeneous) matrix equation

$$P^0 = H_1^0 P^1 \quad (1.32)$$

The set of all 4×4 matrices H of the form (28) is denoted by $E(3)$. A set of **basic homogeneous transformations** generating $E(3)$ is given by

$$\begin{aligned} \text{Rot}_{x,\alpha} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha & 0 \\ 0 & s_\alpha & c_\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; & \text{Rot}_{y,\theta} &= \begin{bmatrix} c_\theta & 0 & s_\theta & 0 \\ 0 & 1 & 0 & 0 \\ -s_\theta & 0 & c_\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ \text{Rot}_{z,\psi} &= \begin{bmatrix} c_\psi & -s_\psi & 0 & 0 \\ s_\psi & c_\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; & \text{Trans}_{x,a} &= \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ \text{Trans}_{y,b} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; & \text{Trans}_{z,c} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Example 1.3 The homogeneous transformation matrix H that represents a rotation of α degrees about the current x -axis followed by a translation of b units along the current x -axis, followed by a translation of d units along the current z -axis, followed by a rotation of θ degrees about the current z -axis, is given by

$$\begin{aligned} H &= \text{Rot}_{x,\alpha} \text{Trans}_{x,b} \text{Trans}_{z,d} \text{Rot}_{z,\theta} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha & 0 \\ 0 & s_\alpha & c_\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & -s_\theta & 0 & 0 \\ s_\theta & c_\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c_\theta & -s_\theta & 0 & b \\ c_\alpha s_\theta & c_\alpha c_\theta & -s_\alpha & -s_\alpha d \\ s_\alpha s_\theta & s_\alpha c_\theta & c_\alpha & c_\alpha d \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned} \quad (1.33)$$

◇

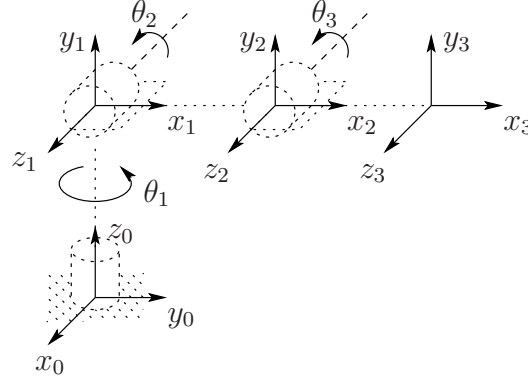


Figure 1.6: Coordinate frames attached to elbow manipulator.

1.3 Forward Kinematics

The forward kinematics problem is concerned with the relationship between the individual joints of the robot manipulator and the position and orientation of the tool or end-effector. Stated more formally, the forward kinematics problem is to determine the position and orientation of the end-effector, given the values for the joint variables of the robot. The joint variables are the angles between the links in the case of revolute or rotational joints, and the link extension in the case of prismatic or sliding joints.

With the assumption that each joint has a single degree-of-freedom, the action of each joint can be described by a single real number: the angle of rotation in the case of a revolute joint or the displacement in the case of a prismatic joint. The objective of forward kinematic analysis is to determine the *cumulative* effect of the entire set of joint variables.

A robot manipulator with n joints will have $n + 1$ links, since each joint connects two links. We number the joints from 1 to n , and we number the links from 0 to n , starting from the base. By this convention, joint i connects link $i - 1$ to link i . We will consider the location of joint i to be fixed with respect to link $i - 1$. *When joint i is actuated, link i moves.* Therefore, link 0 (the first link) is fixed, and does not move when the joints are actuated. Of course the robot manipulator could itself be mobile (e.g., it could be mounted on a mobile platform or on an autonomous vehicle), but we will not consider this case in the present chapter, since it can be handled easily by slightly extending the techniques presented here.

With the i^{th} joint, we associate a *joint variable*, denoted by q_i . In the case of a revolute joint, q_i is the angle of rotation, and in the case of a prismatic joint, q_i is the joint displacement:

$$q_i = \begin{cases} \theta_i & : \text{ joint } i \text{ revolute} \\ d_i & : \text{ joint } i \text{ prismatic} \end{cases} \quad (1.34)$$

To perform the kinematic analysis, we rigidly attach a coordinate frame to each link. In particular, we attach $o_i x_i y_i z_i$ to link i . This means that, whatever motion the robot executes, the coordinates of each point on link i are constant when expressed in the i^{th} coordinate frame. Furthermore, when joint i is actuated, link i and its attached frame, $o_i x_i y_i z_i$, experience a resulting motion. The frame $o_0 x_0 y_0 z_0$, which is attached to the robot base, is referred to as the inertial frame. Figure 6 illustrates the idea of attaching frames rigidly to links in the case of an elbow manipulator.

Now suppose A_i is the homogeneous transformation matrix that expresses the position and orientation of $o_i x_i y_i z_i$ with respect to $o_{i-1} x_{i-1} y_{i-1} z_{i-1}$. The matrix A_i is not constant, but varies as the configuration of the robot is changed. However, the assumption that all joints are either revolute or prismatic means that A_i is a function of only a single joint variable, namely q_i . In other words,

$$A_i = A_i(q_i). \quad (1.35)$$

Now the homogeneous transformation matrix that expresses the position and orientation of $o_j x_j y_j z_j$ with respect to $o_i x_i y_i z_i$ is denoted by T_j^i and is given by

$$\begin{aligned} T_j^i &= A_{i+1} A_{i+2} \dots A_{j-1} A_j \text{ if } i < j \\ T_j^i &= I \text{ if } i = j \\ T_j^i &= (T_i^j)^{-1} \text{ if } j > i. \end{aligned} \quad (1.36)$$

By the manner in which we have rigidly attached the various frames to the corresponding links, it follows that the position of any point on the end-effector, when expressed in frame n , is a constant independent of the configuration of the robot. Denote the position and orientation of the end-effector with respect to the inertial or base frame by a three-vector o_n^0 (which gives the coordinates of the origin of the end-effector frame with respect to the base frame) and the 3×3 rotation matrix R_n^0 , and define the homogeneous transformation matrix

$$H = \begin{bmatrix} R_n^0 & o_n^0 \\ 0 & 1 \end{bmatrix}. \quad (1.37)$$

Then the position and orientation of the end-effector in the inertial frame are given by

$$H = T_n^0 = A_1(q_1) \dots A_n(q_n). \quad (1.38)$$

Each homogeneous transformation A_i is of the form

$$A_i = \begin{bmatrix} R_i^{i-1} & o_i^{i-1} \\ 0 & 1 \end{bmatrix}. \quad (1.39)$$

Hence

$$T_j^i = A_{i+1} \dots A_j = \begin{bmatrix} R_j^i & o_j^i \\ 0 & 1 \end{bmatrix}. \quad (1.40)$$

The matrix R_j^i expresses the orientation of $o_j x_j y_j z_j$ relative to $o_i x_i y_i z_i$ and is given by the rotational parts of the A -matrices as

$$R_j^i = R_{i+1}^i \dots R_j^{j-1}. \quad (1.41)$$

The coordinate vectors o_j^i are given recursively by the formula

$$o_j^i = o_{j-1}^i + R_{j-1}^i o_j^{j-1}, \quad (1.42)$$

In principle, that is all there is to forward kinematics! Determine the functions $A_i(q_i)$, and multiply them together as needed. However, it is possible to achieve a considerable amount of streamlining and simplification by introducing further conventions, such as the Denavit-Hartenberg representation of a joint, and this is the objective of the remainder of the chapter.

Table 1.1: The four DH parameters.

a_i :	link length
α_i :	link twist
d_i :	link offset
θ_i :	joint angle

1.4 Denavit Hartenberg Representation

While it is possible to carry out all of the analysis in this chapter using an arbitrary frame attached to each link, it is helpful to be systematic in the choice of these frames. A commonly used convention for selecting frames of reference in robotic applications is the Denavit-Hartenberg, or D-H convention. In this convention, each homogeneous transformation A_i is represented as a product of four basic transformations

$$\begin{aligned}
 A_i &= R_{z,\theta_i} \text{Trans}_{z,d_i} \text{Trans}_{x,a_i} R_{x,\alpha_i} \\
 &= \begin{bmatrix} c\theta_i & -s\theta_i & 0 & 0 \\ s\theta_i & c\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\alpha_i & -s\alpha_i & 0 \\ 0 & s\alpha_i & c\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
 \end{aligned} \tag{1.43}$$

where the four quantities θ_i , a_i , d_i , α_i are parameters associated with link i and joint i . The four parameters a_i , α_i , d_i , and θ_i in (43) are generally given the names **link length**, **link twist**, **link offset**, and **joint angle**, respectively. These names, summarized in Table 1, derive from specific aspects of the geometric relationship between two coordinate frames, as will become apparent below. Since the matrix A_i is a function of a single variable, it turns out that three of the above four quantities are constant for a given link, while the fourth parameter, θ_i for a revolute joint and d_i for a prismatic joint, is the joint variable.

We know that an arbitrary homogeneous transformation matrix can be characterized by six numbers, such as, for example, three numbers to specify the fourth column of the matrix and three Euler angles to specify the upper left 3×3 rotation matrix. In the D-H representation, in contrast, there are only *four* parameters. How is this possible? The answer is that, while frame i is required to be rigidly attached to link i , we have considerable freedom in choosing the origin and the coordinate axes of the frame. For example, it is not necessary that the origin, o_i , of frame i be placed at the physical end of link i . In fact, it is not even necessary that frame i be placed within the physical link; frame i could lie in free space — so long as frame i is *rigidly attached* to link i . By a clever choice of the origin and the coordinate axes, it is possible to cut down the number of parameters needed from six to four (or even fewer in some cases).

Now that we have established that each homogeneous transformation matrix satisfying conditions (DH1) and (DH2) above can be represented in the form (43), we can in fact give a physical interpretation to each of the four quantities in (43). The parameter a is the distance between the axes z_0 and z_1 , and is measured along the axis x_1 . The angle α is the angle between the axes z_0

and z_1 , measured in a plane normal to x_1 . The positive sense for α is determined from z_0 to z_1 by the right-hand rule as shown in Figure 7. The parameter d is the distance between the origin

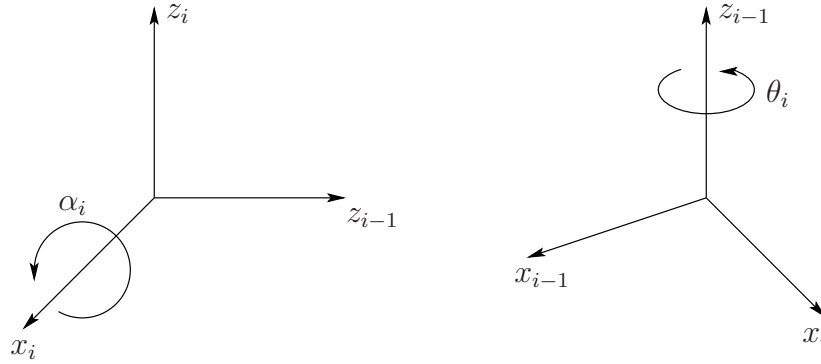


Figure 1.7: Positive sense for α_i and θ_i .

o_0 and the intersection of the x_1 axis with z_0 measured along the z_0 axis. Finally, θ is the angle between x_0 and x_1 measured in a plane normal to z_0 . These physical interpretations will prove useful in developing a procedure for assigning coordinate frames that satisfy the constraints (DH1) and (DH2), and we now turn our attention to developing such a procedure.

1.4.1 Assigning the coordinate frames

For a given robot manipulator, one can always choose the frames $0, \dots, n$ in such a way that the above two conditions are satisfied. In certain circumstances, this will require placing the origin o_i of frame i in a location that may not be intuitively satisfying, but typically this will not be the case. In reading the material below, it is important to keep in mind that the choices of the various coordinate frames are not unique, even when constrained by the requirements above. Thus, it is possible that different engineers will derive differing, but equally correct, coordinate frame assignments for the links of the robot. It is very important to note, however, that the end result (i.e., the matrix T_n^0) will be the same, regardless of the assignment of intermediate link frames (assuming that the coordinate frames for link n coincide). We will begin by deriving the general procedure. We will then discuss various common special cases where it is possible to further simplify the homogeneous transformation matrix.

To start, note that the choice of z_i is arbitrary. By choosing α_i and θ_i appropriately, we can obtain any arbitrary direction for z_i . Thus, for our first step, we assign the axes z_0, \dots, z_{n-1} in an intuitively pleasing fashion. Specifically, we assign z_i to be the axis of actuation for joint $i + 1$. Thus, z_0 is the axis of actuation for joint 1, z_1 is the axis of actuation for joint 2, etc. There are two cases to consider: (i) if joint $i + 1$ is revolute, z_i is the axis of revolution of joint $i + 1$; (ii) if joint $i + 1$ is prismatic, z_i is the axis of translation of joint $i + 1$.

Once we have established the z -axes for the links, we establish the base frame. The choice of a base frame is nearly arbitrary. We may choose the origin o_0 of the base frame to be any point on z_0 . We then choose x_0, y_0 in any convenient manner so long as the resulting frame is right-handed. This sets up frame 0.

Once frame 0 has been established, we begin an iterative process in which we define frame i using frame $i - 1$, beginning with frame 1. Figure 8 will be useful for understanding the process that we now describe.

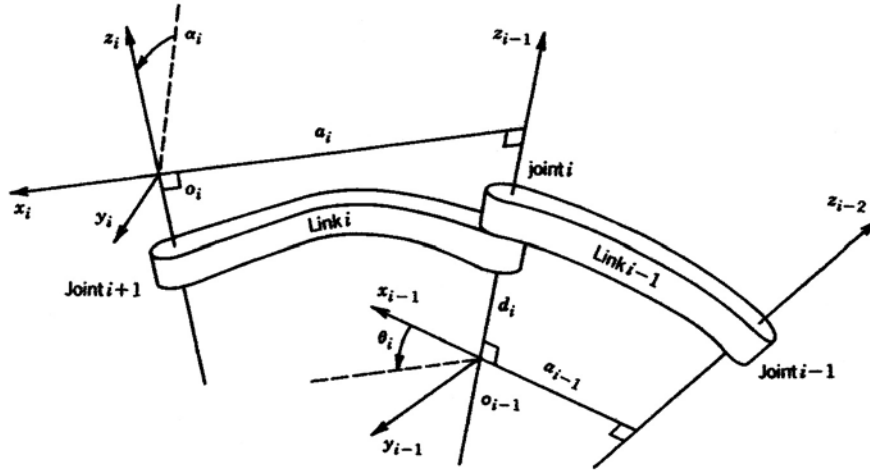


Figure 1.8: Denavit-Hartenberg frame assignment.

In order to set up frame i it is necessary to consider three cases: (i) the axes z_{i-1} , z_i are not coplanar, (ii) the axes z_{i-1} , z_i intersect (iii) the axes z_{i-1} , z_i are parallel. Note that in both cases (ii) and (iii) the axes z_{i-1} and z_i are coplanar. This situation is in fact quite common, as we will see in Section 0.5. We now consider each of these three cases.

(i) z_{i-1} and z_i are not coplanar: Consider the two lines l_{i-1} and l_i that contain the axes z_{i-1} and z_i , respectively. If z_{i-1} and z_i are not coplanar, then there exists a *unique* line segment perpendicular to both l_{i-1} and l_i such that it connects both lines and it has minimum length. The line containing this common normal to z_{i-1} and z_i defines x_i , and the point where this line intersects z_i is the origin o_i . By construction, both conditions (DH1) and (DH2) are satisfied and the vector from o_{i-1} to o_i is a linear combination of z_{i-1} and x_i . The specification of frame i is completed by choosing the axis y_i to form a right-hand frame. Since assumptions (DH1) and (DH2) are satisfied the homogeneous transformation matrix A_i is of the form (43).

(ii) z_{i-1} is parallel to z_i : If the axes z_{i-1} and z_i are parallel, then there are infinitely many common normals between them and condition (DH1) does not specify x_i completely. In this case we are free to choose the origin o_i anywhere along z_i . One often chooses o_i to simplify the resulting equations. The axis x_i is then chosen either to be directed from o_i toward z_{i-1} , along the common normal, or as the opposite of this vector. A common method for choosing o_i is to choose the normal that passes through o_{i-1} as the x_i axis; o_i is then the point at which this normal intersects z_i . In this case, d_i would be equal to zero. Once x_i is fixed, y_i is determined, as usual by the right hand rule. Since the axes z_{i-1} and z_i are parallel, α_i will be zero in this case.

(iii) z_{i-1} intersects z_i : In this case x_i is chosen normal to the plane formed by z_i and z_{i-1} . The positive direction of x_i is arbitrary. The most natural choice for the origin o_i in this case is at the point of intersection of z_i and z_{i-1} . However, any convenient point along the axis z_i suffices. Note that in this case the parameter a_i equals 0.

This constructive procedure works for frames $0, \dots, n-l$ in an n -link robot. To complete the construction, it is necessary to specify frame n . The final coordinate system $o_n x_n y_n z_n$ is commonly

referred to as the **end-effector** or **tool frame** (see Figure 9). The origin o_n is most often placed

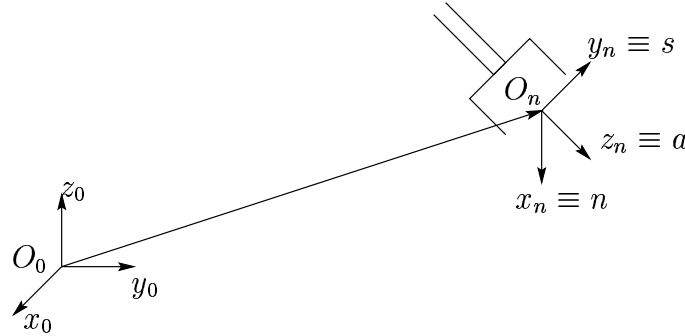


Figure 1.9: Tool frame assignment.

symmetrically between the fingers of the gripper. The unit vectors along the x_n , y_n , and z_n axes are labeled as \mathbf{n} , \mathbf{s} , and \mathbf{a} , respectively. The terminology arises from fact that the direction \mathbf{a} is the **approach** direction, in the sense that the gripper typically approaches an object along the \mathbf{a} direction. Similarly the \mathbf{s} direction is the **sliding** direction, the direction along which the fingers of the gripper slide to open and close, and \mathbf{n} is the direction **normal** to the plane formed by \mathbf{a} and \mathbf{s} .

In all contemporary robots the final joint motion is a rotation of the end-effector by θ_n and the final two joint axes, z_{n-1} and z_n , coincide. Therefore the transformation between the final two coordinate frames is always a translation along z_{n-1} by a distance d_6 followed (or preceded) by a rotation of θ_6 radians about z_{n-1} . This is an important observation which will simplify the computation of the inverse kinematics in the next chapter.

Finally, note the following important fact. In all cases, whether the joint in question is revolute or prismatic, the quantities a_i and α_i are always constant for all i and are characteristic of the manipulator. If joint i is prismatic, then θ_i is also a constant, while d_i is the i^{th} joint variable. Similarly, if joint i is revolute, then d_i is constant and θ_i is the i^{th} joint variable.

1.4.2 Summary

We may summarize the above procedure based on the D-H convention in the following algorithm for deriving the forward kinematics for any manipulator.

Step 1: Locate and label the joint axes z_0, \dots, z_{n-1} .

Step 2: Establish the base frame. Set the origin anywhere on the z_0 -axis. The x_0 and y_0 axes are chosen conveniently to form a right-hand frame.

For $i = 1, \dots, n - 1$, perform Steps 3 to 5.

Step 3: Locate the origin o_i where the common normal to z_i and z_{i-1} intersects z_i . If z_i intersects z_{i-1} locate o_i at this intersection. If z_i and z_{i-1} are parallel, locate o_i in any convenient position along z_i .

Step 4: Establish x_i along the common normal between z_{i-1} and z_i through o_i , or in the direction normal to the $z_{i-1} - z_i$ plane if z_{i-1} and z_i intersect.

Step 5: Establish y_i to complete a right-hand frame.

Table 1.2: Link parameters for 2-link planar manipulator.

Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	0	0	θ_2^*

* variable

Step 6: Establish the end-effector frame $o_n x_n y_n z_n$. Assuming the n -th joint is revolute, set $z_n = \mathbf{a}$ along the direction z_{n-1} . Establish the origin o_n conveniently along z_n , preferably at the center of the gripper or at the tip of any tool that the manipulator may be carrying. Set $y_n = \mathbf{s}$ in the direction of the gripper closure and set $x_n = \mathbf{n}$ as $\mathbf{s} \times \mathbf{a}$. If the tool is not a simple gripper set x_n and y_n conveniently to form a right-hand frame.

Step 7: Create a table of link parameters a_i , d_i , α_i , θ_i .

a_i = distance along x_i from o_i to the intersection of the x_i and z_{i-1} axes.

d_i = distance along z_{i-1} from o_{i-1} to the intersection of the x_i and z_{i-1} axes. d_i is variable if joint i is prismatic.

α_i = the angle between z_{i-1} and z_i measured about x_i (see Figure 7).

θ_i = the angle between x_{i-1} and x_i measured about z_{i-1} (see Figure 7). θ_i is variable if joint i is revolute.

Step 8: Form the homogeneous transformation matrices A_i by substituting the above parameters into (43).

Step 9: Form $T_n^0 = A_1 \cdots A_n$. This then gives the position and orientation of the tool frame expressed in base coordinates.

1.5 Examples

In the D-H convention the only variable angle is θ , so we simplify notation by writing c_i for $\cos \theta_i$, etc. We also denote $\theta_1 + \theta_2$ by θ_{12} , and $\cos(\theta_1 + \theta_2)$ by c_{12} , and so on. In the following examples it is important to remember that the D-H convention, while systematic, still allows considerable freedom in the choice of some of the manipulator parameters. This is particularly true in the case of parallel joint axes or when prismatic joints are involved.

Example 1.4 Planar Elbow Manipulator

Consider the two-link planar arm of Figure 10. The joint axes z_0 and z_1 are normal to the page. We establish the base frame $o_0 x_0 y_0 z_0$ as shown. The origin is chosen at the point of intersection of the z_0 axis with the page and the direction of the x_0 axis is completely arbitrary. Once the base frame is established, the $o_1 x_1 y_1 z_1$ frame is fixed as shown by the D-H convention, where the origin o_1 has been located at the intersection of z_1 and the page. The final frame $o_2 x_2 y_2 z_2$ is fixed by choosing the origin o_2 at the end of link 2 as shown. The link parameters are shown in Table 2. The A -matrices are determined from (43) as

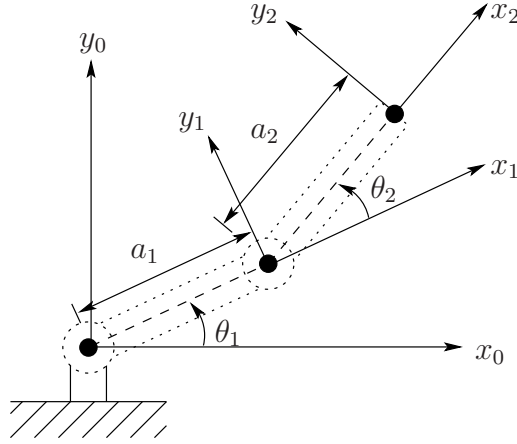


Figure 1.10: Two-link planar manipulator. The z -axes all point out of the page, and are not shown in the figure.

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1 c_1 \\ s_1 & c_1 & 0 & a_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (1.44)$$

$$A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2 c_2 \\ s_2 & c_2 & 0 & a_2 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (1.45)$$

The T -matrices are thus given by

$$T_1^0 = A_1. \quad (1.46)$$

$$T_2^0 = A_1 A_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1 c_1 + a_2 c_{12} \\ s_{12} & c_{12} & 0 & a_1 s_1 + a_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (1.47)$$

Notice that the first two entries of the last column of T_2^0 are the x and y components of the origin o_2 in the base frame; that is,

$$\begin{aligned} x &= a_1 c_1 + a_2 c_{12} \\ y &= a_1 s_1 + a_2 s_{12} \end{aligned} \quad (1.48)$$

are the coordinates of the end-effector in the base frame. The rotational part of T_2^0 gives the orientation of the frame $o_2 x_2 y_2 z_2$ relative to the base frame.

◇

Example 1.5 Three-Link Cylindrical Robot

Consider now the three-link cylindrical robot represented symbolically by Figure 11. We establish o_0 as shown at joint 1. Note that the placement of the origin o_0 along z_0 as well as the direction of the x_0 axis are arbitrary. Our choice of o_0 is the most natural, but o_0 could just as well be placed at joint 2. The axis x_0 is chosen normal to the page. Next, since z_0 and z_1 coincide, the origin o_1

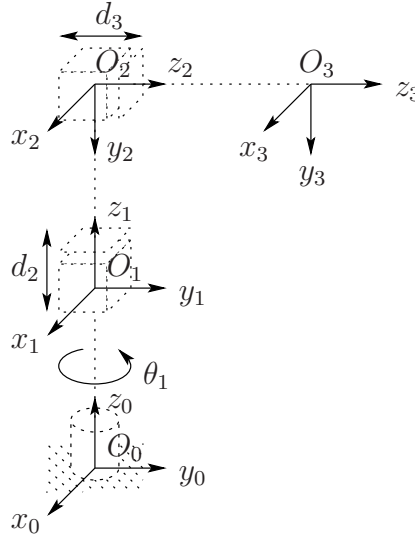


Figure 1.11: Three-link cylindrical manipulator.

Table 1.3: Link parameters for 3-link cylindrical manipulator.

Link	a_i	α_i	d_i	θ_i
1	0	0	d_1	θ_1^*
2	0	-90	d_2^*	0
3	0	0	d_3^*	0

* variable

is chosen at joint 1 as shown. The x_1 axis is normal to the page when $\theta_1 = 0$ but, of course its direction will change since θ_1 is variable. Since z_2 and z_1 intersect, the origin o_2 is placed at this intersection. The direction of x_2 is chosen parallel to x_1 so that θ_2 is zero. Finally, the third frame is chosen at the end of link 3 as shown.

The link parameters are now shown in Table 3. The corresponding A and T matrices are

$$\begin{aligned}
 A_1 &= \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 A_2 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 A_3 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
 \end{aligned} \tag{1.49}$$

$$T_3^0 = A_1 A_2 A_3 = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_3 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (1.50)$$

◇

1.6 Inverse Kinematics

The general problem of inverse kinematics can be stated as follows. Given a 4×4 homogeneous transformation

$$H = \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix} \in E(3) \quad (1.51)$$

with $R \in SO(3)$, find (one or all) solutions of the equation

$$T_n^0(q_1, \dots, q_n) = H \quad (1.52)$$

where

$$T_n^0(q_1, \dots, q_n) = A_1 \cdots A_n. \quad (1.53)$$

Here, H represents the desired position and orientation of the end-effector, and our task is to find the values for the joint variables q_1, \dots, q_n so that $T_n^0(q_1, \dots, q_n) = H$.

Equation (52) results in twelve nonlinear equations in n unknown variables, which can be written as

$$T_{ij}(q_1, \dots, q_n) = h_{ij}, \quad i = 1, 2, 3, \quad j = 1, \dots, 4 \quad (1.54)$$

where T_{ij} , h_{ij} refer to the twelve nontrivial entries of T_n^0 and H , respectively. (Since the bottom row of both T_n^0 and H are $(0,0,0,1)$, four of the sixteen equations represented by (52) are trivial.)

1.6.1 Kinematic Decoupling

Although the general problem of inverse kinematics is quite difficult, it turns out that for manipulators having six joints, with the last three joints intersecting at a point (such as the Stanford Manipulator above), it is possible to decouple the inverse kinematics problem into two simpler problems, known respectively, as **inverse position kinematics**, and **inverse orientation kinematics**. To put it another way, for a six-DOF manipulator with a spherical wrist, the inverse kinematics problem may be separated into two simpler problems, namely first finding the position of the intersection of the wrist axes, hereafter called the **wrist center**, and then finding the orientation of the wrist.

For concreteness let us suppose that there are exactly six degrees-of-freedom and that the last three joint axes intersect at a point O_c . We express (52) as two sets of equations representing the rotational and positional equations

$$R_6^0(q_1, \dots, q_6) = R \quad (1.55)$$

$$o_6^0(q_1, \dots, q_6) = o \quad (1.56)$$

where o and R are the desired position and orientation of the tool frame, expressed with respect to the world coordinate system. Thus, we are given o and R , and the inverse kinematics problem is to solve for q_1, \dots, q_6 .

Now assumption of a spherical wrist means that the axes z_4 , z_5 , and z_6 intersect at o_c and hence the origins o_4 and o_5 assigned by the DH-convention will always be at the wrist center o_c . Often o_3 will also be at o_c , but this is not necessary for our subsequent development. The important point of this assumption for the inverse kinematics is that motion of the final three links about these axes will not change the position of o_c . The position of the wrist center is thus a function of only the first three joint variables. Since the origin of the tool frame (whose desired coordinates are given by o) is simply obtained by a translation of distance d_6 along z_5 from o_c we have

$$o = o_c^0 + d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \quad (1.57)$$

Recall that the third column of R expresses the direction of z_6 with respect to the base frame, and in our case, z_5 and z_6 are the same axis. Thus in order to have the end-effector of the robot at the point with coordinates given by o and with the orientation of the end-effector given by $R = (r_{ij})$, it is necessary and sufficient that the wrist center o_c have coordinates given by

$$o_c^0 = o - d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \quad (1.58)$$

and that the orientation of the frame $o_6x_6y_6z_6$ with respect to the base be given by R . If the components of the end-effector position o are denoted O_x, O_y, O_z and the components of the wrist center o_c^0 are denoted x_c, y_c, z_c then (58) gives the relationship

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix}. \quad (1.59)$$

Using Equation (59) we may find the values of the first three joint variables. This determines the orientation transformation R_3^0 which depends only on these first three joint variables. We can now determine the orientation of the end-effector relative to the frame $o_3x_3y_3z_3$ from the expression

$$R = R_3^0 R_6^3 \quad (1.60)$$

as

$$R_6^3 = (R_3^0)^{-1} R = (R_3^0)^T R. \quad (1.61)$$

The final three joint angles can then be found as a set of Euler angles corresponding to R_6^3 . Note that the right hand side of (61) is completely known since R is given and R_3^0 can be calculated once the first three joint variables are known. The idea of kinematic decoupling is illustrated in Figure 12.

1.6.2 Summary

For this class of manipulators the determination of the inverse kinematics can be summarized by the following algorithm.

Step 1: Find q_1, q_2, q_3 such that the wrist center O_c has coordinates given by

$$o_c^0 = o - d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \quad (1.62)$$

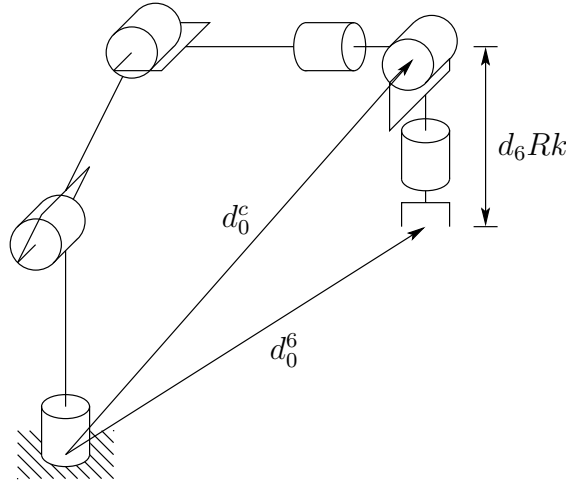


Figure 1.12: Kinematic decoupling.

Step 2: Using the joint variables determined in Step 1, evaluate R_3^0 .

Step 3: Find a set of Euler angles corresponding to the rotation matrix

$$R_6^3 = (R_3^0)^{-1}R = (R_3^0)^T R. \quad (1.63)$$

1.7 Inverse Position: A Geometric Approach

For the common kinematic arrangements that we consider, we can use a geometric approach to find the variables, q_1, q_2, q_3 corresponding to O_c^0 given by (58). We restrict our treatment to the geometric approach for two reasons. First, as we have said, most present manipulator designs are usually designed with a spherical wrist. Indeed, it is partly due to the difficulty of the general inverse kinematics problem that manipulator designs have evolved to their present state. Second, there are few techniques that can handle the general inverse kinematics problem for arbitrary configurations. Since the reader is most likely to encounter robot configurations of the type considered here, the added difficulty involved in treating the general case seems unjustified. The reader is directed to the references at the end of the chapter for treatment of the general case.

In general the complexity of the inverse kinematics problem increases with the number of nonzero link parameters. For most manipulators, many of the a_i, d_i are zero, the α_i are 0 or $\pm\pi/2$, etc. In these cases especially, a geometric approach is the simplest and most natural. We will illustrate this with several important examples.

1.7.1 Articulated Configuration

Consider the elbow manipulator shown in Figure 13. With the components of O_c^0 denoted by x_c, y_c, z_c , we project O_c onto the $x_0 - y_0$ plane as shown in Figure 14. We see from this projection that

$$\theta_1 = A \tan(x_c, y_c), \quad (1.64)$$

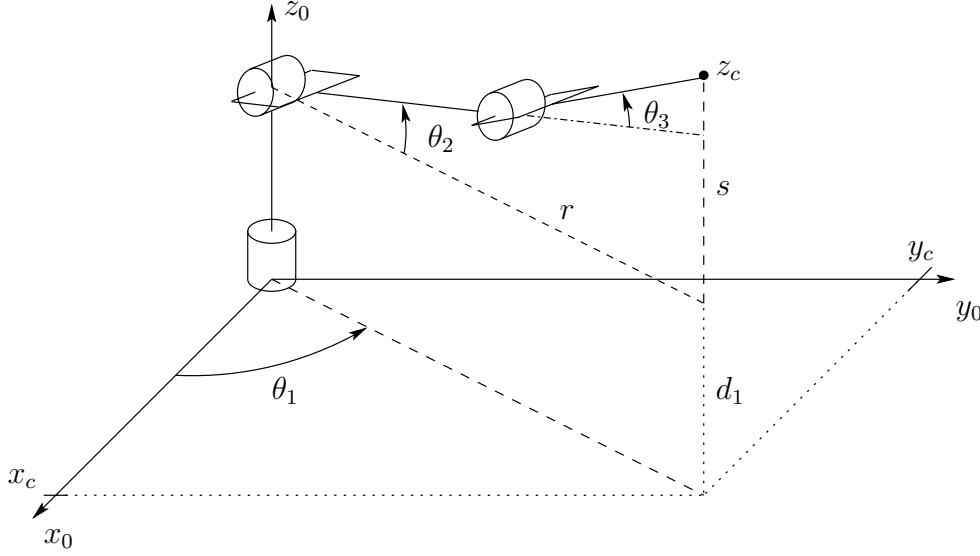
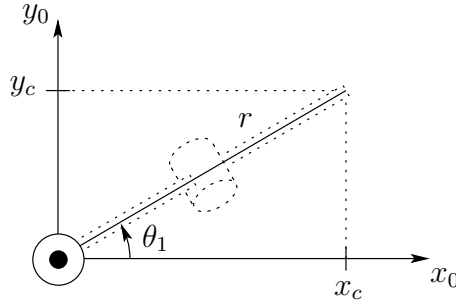


Figure 1.13: Elbow manipulator.

Figure 1.14: Projection of the wrist center onto $x_0 - y_0$ plane.

in which $A \tan(x, y)$ denotes the two argument arctangent function. $A \tan(x, y)$ is defined for all $(x, y) \neq (0, 0)$ and equals the unique angle θ such that

$$\cos \theta = \frac{x}{(x^2 + y^2)^{\frac{1}{2}}}, \quad \sin \theta = \frac{y}{(x^2 + y^2)^{\frac{1}{2}}}. \quad (1.65)$$

For example, $A \tan(1, -1) = -\frac{\pi}{4}$, while $A \tan(-1, 1) = +\frac{3\pi}{4}$.

Note that a second valid solution for θ_1 is

$$\theta_1 = \pi + A \tan(x_c, y_c) \quad (1.66)$$

provided that the solution θ_2 corresponding to (64) is replaced by $\pi - \theta_2$.

These solutions for θ_1 , are valid unless $x_c = y_c = 0$. In this case (64) is undefined and the manipulator is in a singular configuration, shown in Figure 15. In this position the wrist center O_c intersects z_0 ; hence any value of θ_1 leaves O_c fixed. There are thus infinitely many solutions for θ_1 when O_c intersects z_0 .

To find the angles θ_2, θ_3 for the elbow manipulator, given θ_1 , we consider the plane formed by the second and third links as shown in Figure 16. Since the motion of links two and three is planar,

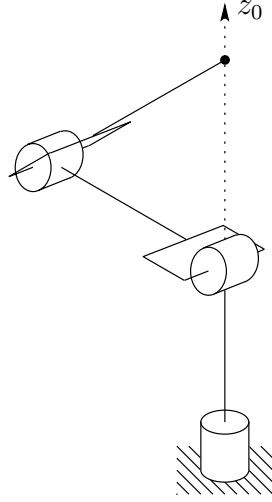


Figure 1.15: Singular configuration.

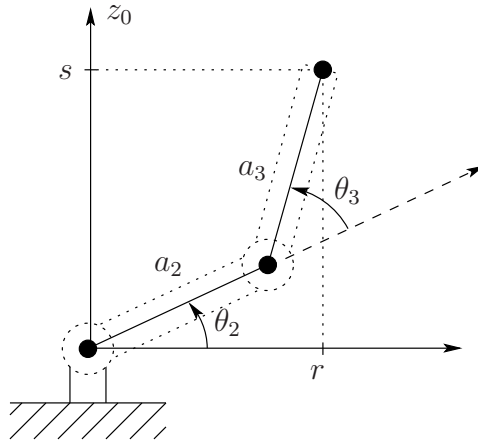


Figure 1.16: Projecting onto the plane formed by links 2 and 3.

the solution is analogous to that of the two-link manipulator and we obtain

$$\begin{aligned} \cos \theta_3 &= \frac{r^2 + s^2 - a_2^2 - a_3^2}{2a_2a_3} \\ &= \frac{x_c^2 + y_c^2 - d^2 + z_c^2 - a_2^2 - a_3^2}{2a_2a_3} := D, \end{aligned} \quad (1.67)$$

since $r^2 = x_c^2 + y_c^2 - d^2$ and $s = z_c$. Hence, θ_3 is given by

$$\theta_3 = A \tan \left(D, \pm \sqrt{1 - D^2} \right). \quad (1.68)$$

Similarly θ_2 is given as

$$\begin{aligned} \theta_2 &= A \tan(r, s) - A \tan(a_2 + a_3 c_3, a_3 s_3) \\ &= A \tan \left(\sqrt{x_c^2 + y_c^2 - d^2}, z_c \right) - A \tan(a_2 + a_3 c_3, a_3 s_3). \end{aligned} \quad (1.69)$$

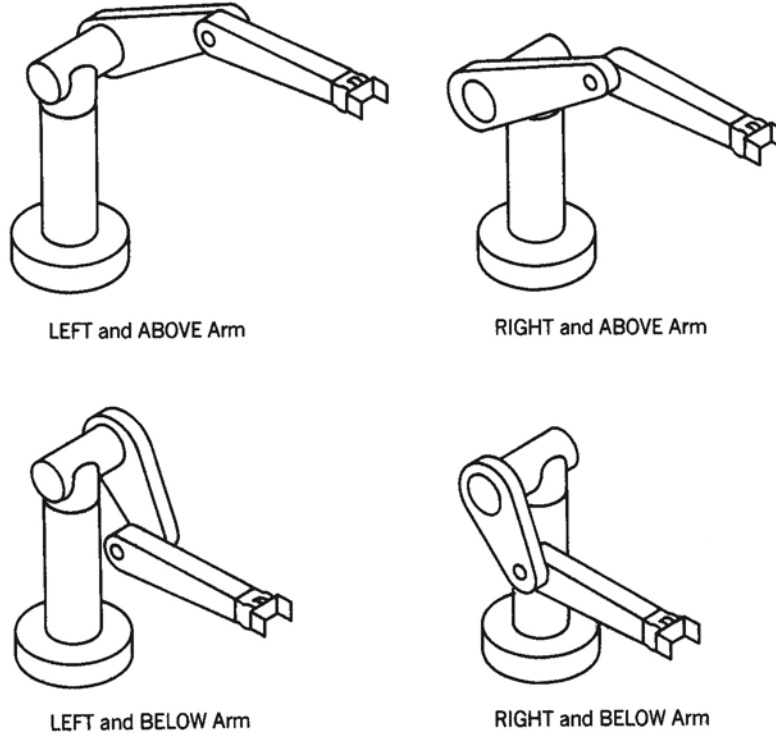


Figure 1.17: Four solutions of the inverse position kinematics for the PUMA manipulator.

The two solutions for θ_3 correspond to the elbow-up position and elbow-down position, respectively.

An example of an elbow manipulator with offsets is the PUMA shown in Figure 17. There are four solutions to the inverse position kinematics as shown. These correspond to the situations left arm-elbow up, left arm-elbow down, right arm-elbow up and right arm-elbow down. We will see that there are two solutions for the wrist orientation thus giving a total of eight solutions of the inverse kinematics for the PUMA manipulator.

1.8 Jacobians

Definition 3 A matrix S is said to be **skew symmetric** if and only if

$$S^T + S = 0. \quad (1.70)$$

Suppose that a rotation matrix R is time varying, so that $R = R(t) \in SO(3)$ for every $t \in \mathbf{R}$. It is easily shown that the time derivative $\dot{R}(t)$ of $R(t)$ is given by

$$\dot{R}(t) = S(t)R(t) \quad (1.71)$$

where the matrix $S(t)$ is skew symmetric. Now, since $S(t)$ is skew symmetric, it can be represented as $S(\omega(t))$ for a unique vector $\omega(t)$. This vector $\omega(t)$ is the **angular velocity** of the rotating frame with respect to the fixed frame at time t . Thus, the time derivative $\dot{R}(t)$ is given by

$$\dot{R}(t) = S(\omega(t))R(t) \quad (1.72)$$

in which $\omega(t)$ is the angular velocity.

1.9 Linear Velocity of a Point Attached to a Moving Frame

We now consider the linear velocity of a point that is rigidly attached to a moving frame. Suppose the point p is rigidly attached to the frame $o_1x_1y_1z_1$, and that $o_1x_1y_1z_1$ is rotating relative to the frame $o_0x_0y_0z_0$. Then the coordinates of p with respect to the frame $o_0x_0y_0z_0$ are given by

$$p^0 = R_1^0(t)p^1. \quad (1.73)$$

The velocity \dot{p}^0 is then given as

$$\dot{p}^0 = \dot{R}_1^0(t)p^1 + R_1^0(t)\dot{p}^1 \quad (1.74)$$

$$\begin{aligned} &= S(\omega^0)R_1^0(t)p^1 \\ &= S(\omega^0)p^0 = \omega^0 \times p^0 \end{aligned} \quad (1.75)$$

which is the familiar expression for the velocity in terms of the vector cross product. Note that (75) follows from that fact that p is rigidly attached to frame $o_1x_1y_1z_1$, and therefore its coordinates relative to frame $o_1x_1y_1z_1$ do not change, giving $\dot{p}^1 = 0$.

Now suppose that the motion of the frame $o_1x_1y_1z_1$ relative to $o_0x_0y_0z_0$ is more general. Suppose that the homogeneous transformation relating the two frames is time-dependent, so that

$$H_1^0(t) = \begin{bmatrix} R_1^0(t) & o_1^0(t) \\ \mathbf{0} & 1 \end{bmatrix}. \quad (1.76)$$

For simplicity we omit the argument t and the subscripts and superscripts on R_1^0 and o_1^0 , and write

$$p^0 = Rp^1 + o. \quad (1.77)$$

Differentiating the above expression using the product rule gives

$$\begin{aligned} \dot{p}^0 &= \dot{R}p^1 + \dot{O} \\ &= S(\omega)Rp^1 + \dot{O} \\ &= \omega \times r + v \end{aligned} \quad (1.78)$$

where $r = Rp^1$ is the vector from O_1 to p expressed in the orientation of the frame $o_0x_0y_0z_0$, and v is the rate at which the origin O_1 is moving.

If the point p is moving relative to the frame $o_1x_1y_1z_1$, then we must add to the term v the term $R(t)\dot{p}^1$, which is the rate of change of the coordinates p^1 expressed in the frame $o_0x_0y_0z_0$.

1.10 Derivation of the Jacobian

Consider an n -link manipulator with joint variables q_1, \dots, q_n . Let

$$T_n^0(\mathbf{q}) = \begin{bmatrix} R_n^0(\mathbf{q}) & o_n^0(\mathbf{q}) \\ \mathbf{0} & 1 \end{bmatrix} \quad (1.79)$$

denote the transformation from the end-effector frame to the base frame, where $\mathbf{q} = (q_1, \dots, q_n)^T$ is the vector of joint variables. As the robot moves about, both the joint variables q_i and the end-effector position o_n^0 and orientation R_n^0 will be functions of time. The objective of this section

is to relate the linear and angular velocity of the end-effector to the vector of joint velocities $\dot{\mathbf{q}}(t)$. Let

$$S(\omega_n^0) = \dot{R}_n^0 (R_n^0)^T \quad (1.80)$$

define the angular velocity vector ω_n^0 of the end-effector, and let

$$v_n^0 = \dot{O}_n^0 \quad (1.81)$$

denote the linear velocity of the end effector. We seek expressions of the form

$$v_n^0 = J_v \dot{\mathbf{q}} \quad (1.82)$$

$$\omega_n^0 = J_\omega \dot{\mathbf{q}} \quad (1.83)$$

where J_v and J_ω are $3 \times n$ matrices. We may write (82) and (83) together as

$$\begin{bmatrix} v_n^0 \\ \omega_n^0 \end{bmatrix} = J_n^0 \dot{\mathbf{q}} \quad (1.84)$$

where J_n^0 is given by

$$J_n^0 = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix}. \quad (1.85)$$

The matrix J_n^0 is called the **Manipulator Jacobian** or **Jacobian** for short. Note that J_n^0 is a $6 \times n$ matrix where n is the number of links. We next derive a simple expression for the Jacobian of any manipulator.

1.10.1 Angular Velocity

We can determine the angular velocity of the end-effector relative to the base by expressing the angular velocity contributed by each joint in the orientation of the base frame and then summing these.

If the i -th joint is revolute, then the i -th joint variable q_i equals θ_i and the axis of rotation is z_{i-1} . Following the convention that we introduced above, let ω_i^{i-1} represent the angular velocity of link i that is imparted by the rotation of joint i , expressed relative to frame $o_{i-1}x_{i-1}y_{i-1}z_{i-1}$. This angular velocity is expressed in the frame $i-1$ by

$$\omega_i^{i-1} = \dot{q}_i z_{i-1}^{i-1} = \dot{q}_i \mathbf{k} \quad (1.86)$$

in which, as above, \mathbf{k} is the unit coordinate vector $(0, 0, 1)^T$.

If the i -th joint is prismatic, then the motion of frame i relative to frame $i-1$ is a translation and

$$\omega_i^{i-1} = 0. \quad (1.87)$$

Thus, if joint i is prismatic, the angular velocity of the end-effector does not depend on q_i , which now equals d_i .

Therefore, the overall angular velocity of the end-effector, ω_n^0 , in the base frame is determined as

$$\begin{aligned} \omega_n^0 &= \rho_1 \dot{q}_1 \mathbf{k} + \rho_2 \dot{q}_2 R_1^0 \mathbf{k} + \cdots + \rho_n \dot{q}_n R_{n-1}^0 \mathbf{k} \\ &= \sum_{i=1}^n \rho_i \dot{q}_i z_{i-1}^0 \end{aligned} \quad (1.88)$$

in which ρ_i is equal to 1 if joint i is revolute and 0 if joint i is prismatic, since

$$z_{i-1}^0 = R_{i-1}^0 \mathbf{k}. \quad (1.89)$$

Of course $z_0^0 = \mathbf{k} = (0, 0, 1)^T$.

The lower half of the Jacobian J_ω , in (85) is thus given as

$$J_\omega = [\rho_1 z_0 \cdots \rho_n z_{n-1}]. \quad (1.90)$$

Note that in this equation, we have omitted the superscripts for the unit vectors along the z-axes, since these are all referenced to the world frame. In the remainder of the chapter we will follow this convention when there is no ambiguity concerning the reference frame.

1.10.2 Linear Velocity

The linear velocity of the end-effector is just \dot{O}_n^0 . By the chain rule for differentiation

$$\dot{O}_n^0 = \sum_{i=1}^n \frac{\partial o_n^0}{\partial q_i} \dot{q}_i. \quad (1.91)$$

Thus we see that the i -th column of $J_{\mathbf{v}}$, which we denote as $J_{\mathbf{v}_i}$ is given by

$$J_{\mathbf{v}_i} = \frac{\partial o_n^0}{\partial q_i}. \quad (1.92)$$

Furthermore this expression is just the linear velocity of the end-effector that would result if \dot{q}_i were equal to one and the other \dot{q}_j were zero. In other words, the i -th column of the Jacobian can be generated by holding all joints fixed but the i -th and actuating the i -th at unit velocity. We now consider the two cases (prismatic and revolute joints) separately.

(i) Case 1: Prismatic Joints

If joint i is prismatic, then it imparts a pure translation to the end-effector. In this case the T_n^0 can be written as the product of three transformations as follows

$$\begin{bmatrix} R_n^0 & o_n^0 \\ \mathbf{0} & 1 \end{bmatrix} = T_n^0 \quad (1.93)$$

$$= T_{i-1}^0 T_i^{i-1} T_n^i \quad (1.94)$$

$$= \begin{bmatrix} R_{i-1}^0 & o_{i-1}^0 \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} R_i^{i-1} & o_i^{i-1} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} R_n^i & o_n^i \\ \mathbf{0} & 1 \end{bmatrix} \quad (1.95)$$

$$= \begin{bmatrix} R_n^0 & R_i^0 o_n^i + R_{i-1}^0 o_i^{i-1} + o_{i-1}^0 \\ \mathbf{0} & 1 \end{bmatrix}, \quad (1.96)$$

which gives

$$o_n^0 = R_i^0 o_n^i + R_{i-1}^0 o_i^{i-1} + o_{i-1}^0. \quad (1.97)$$

If only joint i is allowed to move, then both of o_n^i and o_{i-1}^0 are constant. Furthermore, if joint i is prismatic, then the rotation matrix R_{i-1}^0 is also constant (again, assuming that only joint i

is allowed to move). Finally, recall that, by the DH convention, $o_i^{i-1} = (a_i c_i, a_i s_i, d_i)^T$. Thus, differentiation of o_n^0 gives

$$\frac{\partial o_n^0}{\partial q_i} = \frac{\partial}{\partial d_i} R_{i-1}^0 o_i^{i-1} \quad (1.98)$$

$$= R_{i-1}^0 \frac{\partial}{\partial d_i} \begin{bmatrix} a_i c_i \\ a_i s_i \\ d_i \end{bmatrix} \quad (1.99)$$

$$= \dot{d}_i R_{i-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (1.100)$$

$$= \dot{d}_i z_{i-1}^0, \quad (1.101)$$

in which d_i is the joint variable for prismatic joint i . Thus, (again, dropping the zero superscript on the z-axis) for the case of prismatic joints we have

$$J\mathbf{v}_i = z_{i-1}. \quad (1.102)$$

(ii) Case 2: Revolute Joints

If joint i is revolute, then we have $q_i = \theta_i$. Starting with (97), and letting $q_i = \theta_i$, since R_i^0 is not constant with respect to θ_i , we obtain

$$\frac{\partial}{\partial \theta_i} o_n^0 = \frac{\partial}{\partial \theta_i} [R_i^0 o_n^i + R_{i-1}^0 o_i^{i-1}] \quad (1.103)$$

$$= \frac{\partial}{\partial \theta_i} R_i^0 o_n^i + R_{i-1}^0 \frac{\partial}{\partial \theta_i} o_i^{i-1} \quad (1.104)$$

$$= \dot{\theta}_i S(z_{i-1}^0) R_i^0 o_n^i + \dot{\theta}_i S(z_{i-1}^0) R_{i-1}^0 o_i^{i-1} \quad (1.105)$$

$$= \dot{\theta}_i S(z_{i-1}^0) [R_i^0 o_n^i + R_{i-1}^0 o_i^{i-1}] \quad (1.106)$$

$$= \dot{\theta}_i S(z_{i-1}^0) (o_n^0 - o_{i-1}^0) \quad (1.107)$$

$$= \dot{\theta}_i z_{i-1}^0 \times (o_n^0 - o_{i-1}^0). \quad (1.108)$$

The second term in (105) is derived as follows:

$$R_{i-1}^0 \frac{\partial}{\partial \theta_i} \begin{bmatrix} a_i c_i \\ a_i s_i \\ d_i \end{bmatrix} = R_{i-1}^0 \begin{bmatrix} -a_i s_i \\ a_i c_i \\ 0 \end{bmatrix} \dot{\theta}_i \quad (1.109)$$

$$= R_{i-1}^0 S(\mathbf{k} \dot{\theta}_i) o_i^{i-1} \quad (1.110)$$

$$= R_{i-1}^0 S(\mathbf{k} \dot{\theta}_i) (R_{i-1}^0)^T R_{i-1}^0 o_i^{i-1} \quad (1.111)$$

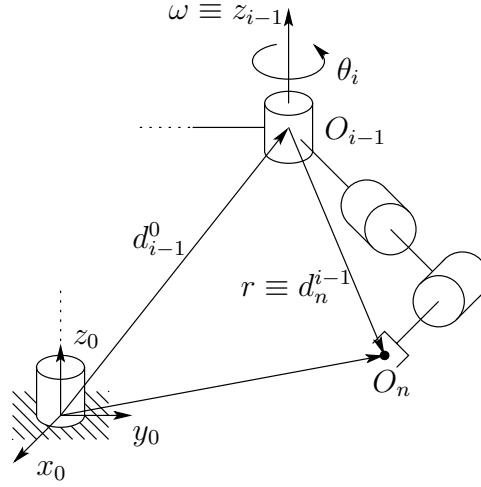
$$= S(R_{i-1}^0 \mathbf{k} \dot{\theta}_i) R_{i-1}^0 o_i^{i-1} \quad (1.112)$$

$$= \dot{\theta}_i S(z_{i-1}^0) R_{i-1}^0 o_i^{i-1}. \quad (1.113)$$

Equation (110) follows by straightforward computation. Thus

$$J\mathbf{v}_i = z_{i-1} \times (O_n - O_{i-1}), \quad (1.114)$$

in which we have, following our convention, omitted the zero superscripts. Figure 18 illustrates a second interpretation of (114). As can be seen in the figure, $O_n - O_{i-1} = \mathbf{r}$ and $z_{i-1} = \omega$ in the familiar expression $\mathbf{v} = \omega \times \mathbf{r}$.

Figure 1.18: Motion of the end-effector due to link i .

Combining the Angular and Linear Jacobians

As we have seen in the preceding section, the upper half of the Jacobian $J_{\mathbf{v}}$ is given as

$$J_{\mathbf{v}} = [J_{\mathbf{v}_1} \cdots J_{\mathbf{v}_n}] \quad (1.115)$$

where the i -th column $J_{\mathbf{v}_i}$ is

$$J_{\mathbf{v}_i} = z_{i-1} \times (O_n - O_{i-1}) \quad (1.116)$$

if joint i is revolute and

$$J_{\mathbf{v}_i} = z_{i-1} \quad (1.117)$$

if joint i is prismatic.

The lower half of the Jacobian is given as

$$J_{\omega} = [J_{\omega_1} \cdots J_{\omega_n}] \quad (1.118)$$

where the i -th column J_{ω_i} is

$$J_{\omega_i} = z_{i-1} \quad (1.119)$$

if joint i is revolute and

$$J_{\omega_i} = 0 \quad (1.120)$$

if joint i is prismatic.

Now putting the upper and lower halves of the Jacobian together we have shown that the Jacobian for an n -link manipulator is of the form

$$J = [J_1 J_2 \cdots J_n] \quad (1.121)$$

where the i -th column J_i is given by

$$J_i = \begin{bmatrix} z_{i-1} \times (O_n - O_{i-1}) \\ z_{i-1} \end{bmatrix} \quad (1.122)$$

if joint i is revolute and

$$J_i = \begin{bmatrix} z_{i-1} \\ 0 \end{bmatrix} \quad (1.123)$$

if joint i is prismatic.

The above formulas make the determination of the Jacobian of any manipulator simple since all of the quantities needed are available once the forward kinematics are worked out. Indeed the only quantities needed to compute the Jacobian are the unit vectors z_i and the coordinates of the origins O_1, \dots, O_n . A moment's reflection shows that the coordinates for z_i w.r.t. the base frame are given by the first three elements in the third column of T_i^0 while O_i is given by the first three elements of the fourth column of T_i^0 . Thus only the third and fourth columns of the T matrices are needed in order to evaluate the Jacobian according to the above formulas.

1.11 Examples

Example 1.6 Consider the two-link planar manipulator of Example 0.4. Since both joints are revolute the Jacobian matrix, which in this case is 6×2 , is of the form

$$J(q) = \begin{bmatrix} z_0 \times (O_2 - O_0) & z_1 \times (O_2 - O_1) \\ z_0 & z_1 \end{bmatrix}. \quad (1.124)$$

The various quantities above are easily seen to be

$$O_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad O_1 = \begin{bmatrix} a_1 c_1 \\ a_1 s_1 \\ 0 \end{bmatrix} \quad O_2 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ 0 \end{bmatrix} \quad (1.125)$$

$$z_0 = z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \quad (1.126)$$

Performing the required calculations then yields

$$J = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}. \quad (1.127)$$

◇

Example 1.7 SCARA Manipulator We will now derive the Jacobian of the SCARA manipulator of Example 3.3.6. This Jacobian is a 6×4 matrix since the SCARA has only four degrees-of-freedom. As before we need only compute the matrices $T_j^0 = A_1 \dots A_j$.

Since joints 1, 2, and 4 are revolute and joint 3 is prismatic, and since $O_4 - O_3$ is parallel to z_3 (and thus, $z_3 \times (O_4 - O_3) = 0$), the Jacobian is of the form

$$J = \begin{bmatrix} z_0 \times (O_4 - O_0) & z_1 \times (O_4 - O_1) & z_2 & 0 \\ z_0 & z_1 & 0 & z_3 \end{bmatrix}. \quad (1.128)$$

Performing the indicated calculations, one obtains

$$O_1 = \begin{bmatrix} a_1 c_1 \\ a_1 s_1 \\ 0 \end{bmatrix} \quad O_2 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ 0 \end{bmatrix} \quad (1.129)$$

$$O_4 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ d_3 - d_4 \end{bmatrix}. \quad (1.130)$$

Similarly $z_0 = z_1 = \mathbf{k}$, and $z_2 = z_3 = -\mathbf{k}$. Therefore the Jacobian of the SCARA Manipulator is

$$J = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} & 0 & 0 \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}. \quad (1.131)$$

◇

1.12 Singularities

The $6 \times n$ Jacobian $J(\mathbf{q})$ defines a mapping

$$\dot{\mathbf{X}} = J(\mathbf{q})\dot{\mathbf{q}} \quad (1.132)$$

between the vector $\dot{\mathbf{q}}$ of joint velocities and the vector $\dot{\mathbf{X}} = (\mathbf{v}, \boldsymbol{\omega})^T$ of end-effector velocities. Infinitesimally this defines a linear transformation

$$d\mathbf{X} = J(\mathbf{q})d\mathbf{q} \quad (1.133)$$

between the differentials $d\mathbf{q}$ and $d\mathbf{X}$. These differentials may be thought of as defining directions in \mathbf{R}^6 , and \mathbf{R}^n , respectively.

Since the Jacobian is a function of the configuration \mathbf{q} , those configurations for which the rank of J decreases are of special significance. Such configurations are called **singularities** or **singular configurations**. Identifying manipulator singularities is important for several reasons.

1. Singularities represent configurations from which certain directions of motion may be unattainable.
2. At singularities, bounded end-effector velocities may correspond to unbounded joint velocities.
3. At singularities, bounded end-effector forces and torques may correspond to unbounded joint torques.
4. Singularities usually (but not always) correspond to points on the boundary of the manipulator workspace, that is, to points of maximum reach of the manipulator.
5. Singularities correspond to points in the manipulator workspace that may be unreachable under small perturbations of the link parameters, such as length, offset, etc.
6. Near singularities there will not exist a unique solution to the inverse kinematics problem. In such cases there may be no solution or there may be infinitely many solutions.

1.12.1 Decoupling of Singularities

We have seen that a set of forward kinematic equations can be derived for any manipulator by attaching a coordinate frame rigidly to each link in any manner that we choose, computing a set of homogeneous transformations relating the coordinate frames, and multiplying them together as needed. The D-H convention is merely a systematic way to do this. Although the resulting equations are dependent on the coordinate frames chosen, the manipulator configurations themselves are geometric quantities, independent of the frames used to describe them. Recognizing this fact allows us to decouple the determination of singular configurations, for those manipulators with spherical wrists, into two simpler problems. The first is to determine so-called **arm singularities**, that is, singularities resulting from motion of the arm, which consists of the first three or more links, while the second is to determine the **wrist singularities** resulting from motion of the spherical wrist.

For the sake of argument, suppose that $n = 6$, that is, the manipulator consists of a 3-DOF arm with a 3-DOF spherical wrist. In this case the Jacobian is a 6×6 matrix and a configuration \mathbf{q} is singular if and only if

$$\det J(\mathbf{q}) = 0. \quad (1.134)$$

If we now partition the Jacobian J into 3×3 blocks as

$$J = [J_P \mid J_O] = \left[\begin{array}{c|c} \frac{J_{11}}{J_{21}} & \frac{J_{12}}{J_{22}} \end{array} \right] \quad (1.135)$$

then, since the final three joints are always revolute

$$J_O = \left[\begin{array}{ccc} z_3 \times (O_6 - O_3) & z_4 \times (O_6 - O_4) & z_5 \times (O_6 - O_5) \\ z_3 & z_4 & z_5 \end{array} \right]. \quad (1.136)$$

Since the wrist axes intersect at a common point O , if we choose the coordinate frames so that $O_3 = O_4 = O_5 = O_6 = O$, then J_O becomes

$$J_O = \left[\begin{array}{ccc} 0 & 0 & 0 \\ z_3 & z_4 & z_5 \end{array} \right] \quad (1.137)$$

and the i -th column J_i of J_p is

$$J_i = \left[\begin{array}{c} z_{i-1} \times (O - O_{i-1}) \\ z_{i-1} \end{array} \right] \quad (1.138)$$

if joint i is revolute and

$$J_i = \left[\begin{array}{c} z_{i-1} \\ 0 \end{array} \right] \quad (1.139)$$

if joint i is prismatic. In this case the Jacobian matrix has the block triangular form

$$J = \left[\begin{array}{cc} J_{11} & 0 \\ J_{21} & J_{22} \end{array} \right] \quad (1.140)$$

with determinant

$$\det J = \det J_{11} \det J_{22} \quad (1.141)$$

where J_{11} and J_{22} are each 3×3 matrices. J_{11} has i -th column $z_{i-1} \times (O - O_{i-1})$ if joint i is revolute, and z_{i-1} if joint i is prismatic, while

$$J_{22} = [z_3 \ z_4 \ z_5]. \quad (1.142)$$

Therefore the set of singular configurations of the manipulator is the union of the set of arm configurations satisfying $\det J_{11} = 0$ and the set of wrist configurations satisfying $\det J_{22} = 0$. *Note that this form of the Jacobian does not necessarily give the correct relation between the velocity of the end-effector and the joint velocities.* It is intended only to simplify the determination of singularities.

1.12.2 Wrist Singularities

We can now see from (142) that a spherical wrist is in a singular configuration whenever the vectors z_3 , z_4 and z_5 are linearly dependent. Referring to Figure 19 we see that this happens when the

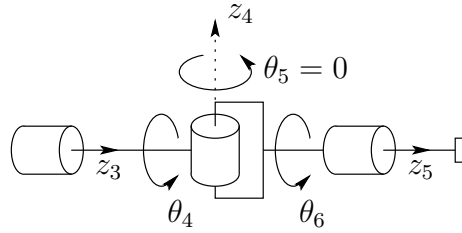


Figure 1.19: Spherical wrist singularity.

joint axes z_3 and z_5 are collinear. In fact, whenever two revolute joint axes anywhere are collinear, a singularity results since an equal and opposite rotation about the axes results in no net motion of the end-effector. This is the only singularity of the spherical wrist, and is unavoidable without imposing mechanical limits on the wrist design to restrict its motion in such a way that z_3 and z_5 are prevented from lining up.

1.12.3 Arm Singularities

In order to investigate arm singularities we need only to compute J_{11} according to (138) and (139), which is the same formula derived previously with the wrist center O in place of O_6 .

Example 1.8 Elbow Manipulator Singularities \diamond Consider the three-link articulated manipulator with coordinate frames attached as shown in Figure 20. It is left as an exercise to show that

$$J_{11} = \begin{bmatrix} -a_2 s_1 c_2 - a_3 s_1 c_{23} & -a_2 s_2 c_1 - a_3 s_{23} c_1 & -a_3 c_1 s_{23} \\ a_2 c_1 c_2 + a_3 c_1 c_{23} & -a_2 s_1 s_2 - a_3 s_1 s_{23} & -a_3 s_1 s_{23} \\ 0 & a_2 c_2 + a_3 c_{23} & a_3 c_{23} \end{bmatrix} \quad (1.143)$$

and that the determinant of J_{11} is

$$\det J_{11} = a_2 a_3 s_3 (a_2 c_2 + a_3 c_{23}). \quad (1.144)$$

We see from (144) that the elbow manipulator is in a singular configuration whenever

$$s_3 = 0, \quad \text{that is, } \theta_3 = 0 \text{ or } \pi \quad (1.145)$$

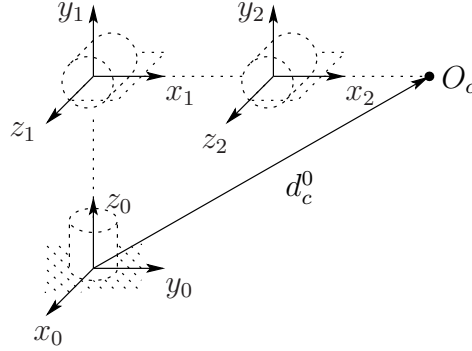


Figure 1.20: Elbow manipulator.

and whenever

$$a_2 c_2 + a_3 c_{23} = 0. \quad (1.146)$$

The situation of (145) is shown in Figure 21 and arises when the elbow is fully extended or fully

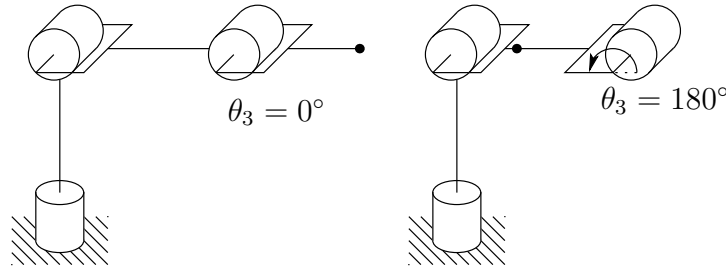


Figure 1.21: Elbow singularities of the elbow manipulator.

retracted as shown. The second situation (146) is shown in Figure 22. This configuration occurs when the wrist center intersects the axis of the base rotation, z_0 .

1.13 Summary

The **forward kinematics map** is a function

$$X_0 = \begin{pmatrix} x(q) \\ R(q) \end{pmatrix} = f_0(q) : \mathcal{T}^n \rightarrow SE(3) \quad (1.147)$$

from configuration space to task space which gives the end-effector pose in terms of the joint configuration. The **inverse kinematics map** gives the joint configuration as a function of the end-effector pose. The forward kinematics map is many-to-one, so that several joint space configurations may give rise to the same end-effector pose. This means that the forward kinematics always has a unique pose for each configuration, while the inverse kinematics has multiple solutions, in general.

The kinematics problem is compounded by the difficulty of parametrizing the rotation group, $SO(3)$. It is well-known that there does not exist a minimal set of coordinates to “cover” $SO(3)$,

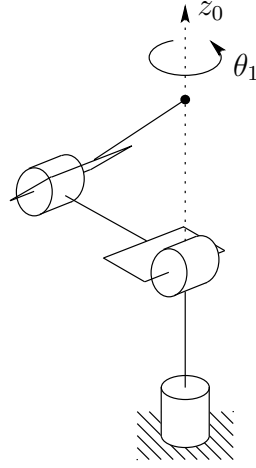


Figure 1.22: Singularity of the elbow manipulator with no offsets.

i.e., a single set of three variables to represent all orientations in $SO(3)$ uniquely. The most common representations used are **Euler angles** and **quaternions**. Representational singularities, which are points at which the representation fails to be unique, give rise to a number of computational difficulties in motion planning and control.

Given a minimal representation for $SO(3)$, for example, a set of Euler angles ϕ, θ, ψ , the forward kinematics map may also be defined by a function

$$X_1 = \begin{pmatrix} x(q) \\ o(q) \end{pmatrix} = f_1(\cdot) : \mathcal{T}^n \rightarrow \mathbb{R}^6 \quad (1.148)$$

where $x(q) \in \mathbb{R}^3$ gives the Cartesian position of the end-effector and $o(q) = (\phi(q), \theta(q), \psi(q))^T$ represents the orientation of the end-effector. The non-uniqueness of the inverse kinematics in this case will include multiplicities due to the particular representation of $SO(3)$ in addition to multiplicities intrinsic to the geometric structure of the manipulator.

Velocity kinematics refers to the relationship between the joint velocities and the end-effector velocities. If the mapping f_0 from (147) is used to represent the forward kinematics, then the velocities are given by

$$V = \begin{pmatrix} v \\ \omega \end{pmatrix} = J_0(q)\dot{q} \quad (1.149)$$

where $J_0(q)$ is a $6 \times n$ matrix, called the **manipulator Jacobian**. The vectors v and ω represent the linear and angular velocity, respectively, of the end-effector. The linear velocity $v \in \mathbb{R}^3$ is just $\frac{d}{dt}x(q)$, where $x(q)$ is the end-effector position vector from (147). It is a little more difficult to see how the angular velocity vector ω is computed since the end-effector orientation in (147) is specified by a matrix $R \in SO(3)$. If $\omega = (\omega_x, \omega_y, \omega_z)^T$ is a vector in \mathbb{R}^3 , we may define a skew-symmetric matrix, $S(\omega)$, according to

$$S(\omega) = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}. \quad (1.150)$$

The set of all skew-symmetric matrices is denoted by $so(3)$. Now, if $R(t)$ belongs to $SO(3)$ for all t , it can be shown that

$$\dot{R} = S(\omega(t))R \quad (1.151)$$

for a unique vector $\omega(t)$. The vector $\omega(t)$ thus defined is the angular velocity of the end-effector frame relative to the base frame.

If the mapping f_1 is used to represent the forward kinematics, then the velocity kinematics is written as

$$\dot{X}_1 = J_1(q)\dot{q} \quad (1.152)$$

where $J_1(q) = \partial f_1 / \partial q$ is the $6 \times n$ Jacobian of the function f_1 . In the sequel we will use J to denote either the matrix J_0 or J_1 .